Journal of Engineering Mathematics, Vol. 16, No. 1, January 1982

On the regularity of the principal value of the double-layer potential

H. SCHIPPERS

*Department of Numerical Analysis, Mathematical Centre, 1098 SJ Amsterdam, The Netherlands**

(Received April 10, 1981)

SUMMARY

In the present paper we discuss the regularity of the principal value of the potential due to a doublet distribution μ along the boundary S of a two-dimensional ($2-D$) open connected set. Assuming S to be a Lyapunov boundary and μ to be essentially bounded, we prove that the principal value in $2-D$ is more regular than the one in $3-D$. This result is applied to the aerodynamics problem of calculation of potential flows around $2-D$ bodies.

1. Introduction

The solution of the Dirichlet problem for Laplace's equation in two and three dimensions can be represented as a double-layer potential, respectively:

$$\phi_d^{(2)}(\zeta) = \frac{1}{2\pi} \int_S \mu(z) \frac{\partial}{\partial n_z} \log |r_{z\zeta}| \, dS_z, \quad \zeta \notin S, \tag{1.1.1}$$

$$\phi_d^{(3)}(\zeta) = \frac{1}{4\pi} \int_S \mu(z) \frac{\partial}{\partial n_z} 1/|r_{z\zeta}| \, dS_z, \quad \zeta \notin S, \tag{1.1.2}$$

where n_z is the outward normal to the surface S at the point z, $r_{z\zeta} = z - \zeta$ and $\mu(\cdot)$ is called the doublet distribution. These potentials are discontinuous across the surface. We denote the principal value of $\phi_d(\zeta)$ by

$$\bar{\phi}_d^{(m)}(\zeta) = \frac{2^{1-m}}{\pi} \int_S \mu(z) \frac{\cos(n_z, z - \zeta)}{|r_{z\zeta}|^{m-1}} \, dS_z, \quad \zeta \in S, \tag{1.2}$$

where $m = 2,3$ for the two- and three-dimensional cases, respectively. Assuming S to be a Lyapunov surface and μ to be essentially bounded Günter [1, p. 49] proves that $\bar{\phi}_d^{(3)} \in H^{0,\alpha}(S)$, where $H^{k,\alpha}(S)$ denotes the class of continuous functions whose derivatives of order k satisfy a uniform Hölder condition with exponent α. In Section 2 of the present paper we prove that in the two-dimensional case $\bar{\phi}_d^{(2)} \in H^{1,\alpha}(S)$, i.e. the principal value is more regular than in the three-dimensional case. This property of regularity has an important application in aerodynamics: the calculation of potential flow around aerofoils.

*Present address: National Aerospace Laboratory, Anthony Fokkerweg, 1059CM Amsterdam, The Netherlands.

022-0833/82/01/059-18$00.20/0

In this problem the doublet distribution μ is the solution of the following Fredholm equation of the second kind:

$$\mu(\zeta) + \frac{1}{\pi} \int_S \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z = -2U \cdot \zeta, \qquad \zeta \in S, \tag{1.3}$$

where U is the velocity vector of the undisturbed flow and $U \cdot \zeta$ denotes the usual inner product in $\mathbb{R}^2$. We write this equation in operator notation as

$$(I-K)\mu = g, \tag{1.4}$$

with $K\mu(\zeta) = -\overline{\phi}_d^{(2)}(\zeta)$ and $g(\zeta) = -2U \cdot \zeta$. The result of the present paper shows that the linear integral operator K maps from the Banach space $L_\infty(S)$ of essentially bounded functions into the Banach space $H^{1,\alpha}(S)$. Moreover, the operator K is a bounded mapping from $L_\infty(S)$ into $H^{1,\alpha}(S)$.

In Section 3 we shall approximate μ by a piecewise constant function μ_N. From the results of Section 2 it follows that $K\mu_N \in H^{1,\alpha}(S)$. We shall use this result to supply error bounds for $\|\mu-\mu_N\|$ and $\max_i |\mu(\zeta_i) - \mu_i|$, where $\zeta_1, \zeta_2, \ldots, \zeta_N$ are the collocation points. This collocation method yields a large linear system of equations. In [7] we have iteratively solved this system by a multiple-grid method. Using $K\mu_N \in H^{1,\alpha}(S)$ we were able to estimate the rate of convergence of the multiple-grid process.

2. Regularity result

First, we give some definitions which have been taken from Günter [1]. Let $D \subset \mathbb{R}^2$ be a bounded simply connected open set with boundary S and closure $\bar{D}$.

Definition 2.1. $C^k(D)$ $(C^k(\bar{D}))$ denotes the class of functions, which are k times continuously differentiable in $D(\bar{D})$.

Definition 2.2. $C^{k,\alpha}(D)$ $(C^{k,\alpha}(\bar{D}))$ denotes the subclass of functions in $C^k(D)$ $(C^k(\bar{D}))$, whose derivatives of order k satisfy a uniform Hölder condition with exponent α, $0 < \alpha < 1$.

Definition 2.3. $L^{k,\alpha}$ $(k \geq 1)$ denotes the class of rectifiable contours S in 2-dimensional Euclidean space with the property that for every point P on S there exists a number $\epsilon > 0$ such that the part Σ of S within the circle $B_{\epsilon,P}$ of radius ϵ and centre P, for some orientation of the axes of the coordinate system (x, y), admits a representation

$$y = F(x), \quad x \in \bar{D}_{\epsilon,P}, \tag{2.1}$$

where $F \in C^{k,\alpha}(\bar{D}_{\epsilon,P})$, $\bar{D}_{\epsilon,P}$ the projection of the part of S within $B_{\epsilon,P}$ on the line $y = 0$.

We give an illustration of Definition 2.3 in the following figure:

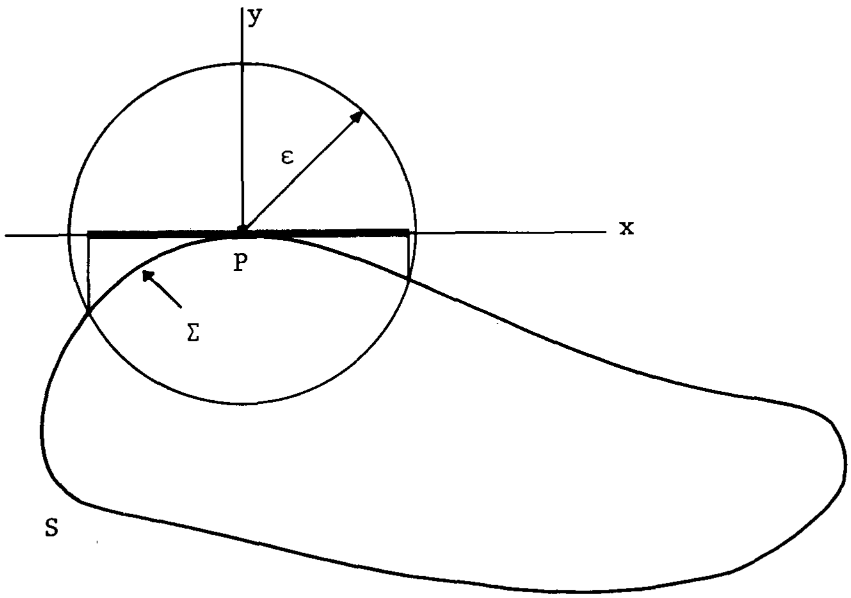

Figure 2.1. Illustration of Definition 2.3 ($\bar{D}_{\epsilon,P}$ is given by ▬▬▬▬▬).

We remark that Definition 2.3 implies that S is bounded.

Definition 2.4. $H^{k,\alpha}(S)$ denotes the class of function f defined on S with the property that the function $\hat{f}$ defined by

$$\hat{f}(x) = f(x, F(x)), \quad x \in \bar{D}_{\epsilon,P},$$

with $F(x)$ and $\bar{D}_{\epsilon,P}$ as in Definition 2.3, belongs to the class $C^{k,\alpha}(\bar{D}_{\epsilon,P})$.

Remark 2.1. Let $S \in L^{k,\alpha}$ with $k \geq 2$ and let $z, \zeta \in S$. Then

$$\lim_{\zeta \to z} \frac{2 \cos (n_z, z-\zeta)}{|z-\zeta|} = \kappa(z),$$

where $\kappa(z)$ is the curvature at z. Moreover, κ belongs to the space $H^{k-2,\alpha}(S)$.

Proof. Let (ξ, η) be a local coordinate system about a certain point $P \in S$ (see Figure 2.1). By Definition 2.3 the points z and ζ may be represented by $(x, F(x))$ and $(\xi, F(\xi))$, respectively. Now

$$\cos (n_z, z-\zeta) = \frac{F(x) - F(\xi) + (\xi - x)F'(x)}{\{(x-\xi)^2 + (F(x) - F(\xi))^2\}^{\frac{1}{2}} \{1 + (F'(x))^2\}^{\frac{1}{2}}},$$

whence

$$\lim_{\zeta \to z} \frac{2 \cos (n_z, z-\zeta)}{|z-\zeta|} = \lim_{\xi \to x} \frac{2\{F(x) - F(\xi) + (\xi - x) F'(x)\}}{\{(x-\xi)^2 + (F(x) - F(\xi))^2\} \{1 + (F'(x))^2\}^{\frac{1}{2}}} .$$

Since

$$F(x) - F(\xi) + (\xi - x)F'(x) = (\xi - x)^2 \int_0^1 t\, F''(x + (\xi - x)\, t)\, \mathrm{d}t$$

and

$$F(x) - F(\xi) = (x - \xi) \int_0^1 F'(x + (\xi - x)\, t)\, \mathrm{d}t,$$

we obtain

$$\lim_{\zeta \to z} \frac{2 \cos (n_z, z - \zeta)}{|z - \zeta|} = F''(x)/\{1 + (F'(x))^2\}^{3/2},$$

which is the definition of the curvature κ. Since $F \in C^{k,\alpha}(\bar{D}_{\epsilon,p})$ it follows that κ has continuous derivatives up to order $k-2$. □

In the two-dimensional case the potential due to a doublet distribution μ along the boundary is given by:

$$\phi_d(\zeta) = \frac{1}{2\pi} \oint_S \mu(z) \frac{\cos (n_z, z - \zeta)}{|z - \zeta|} \, \mathrm{d}S_z, \tag{2.2}$$

with $\zeta \notin S$. The contour integration is taken along the boundary in a counter-clockwise direction. Since $\phi_d = \phi_d^{(2)}$ we further omit the upper index (2).

Lemma 2.1. *Let $S \in L^{2,\alpha}$ and $\mu \in H^{1,\alpha}(S)$. If ζ approaches S we have* (Plemelj-Privalov formulae):

$$\phi_d^+(\zeta) = -\tfrac{1}{2}\mu(\zeta) + \tfrac{1}{2}\bar{\phi}_d(\zeta), \tag{2.2.1}$$

$$\phi_d^-(\zeta) = \tfrac{1}{2}\mu(\zeta) + \tfrac{1}{2}\bar{\phi}_d(\zeta), \tag{2.2.2}$$

with

$$\bar{\phi}_d(\zeta) = \frac{1}{\pi} \oint_S \mu(z) \frac{\cos (n_z, z - \zeta)}{|z - \zeta|} \, \mathrm{d}S_z, \quad \zeta \in S, \tag{2.2.3}$$

where ϕ_d^+ and ϕ_d^- denote the limit from the outer and inner side respectively.

Proof. See Muschelischwili [4, pp. 36–42, p. 52]. □

For $z = \zeta$ the integrand in (2.2.3) is defined by its limit value, i.e. the curvature at ζ. In Lemma 2.1 we assume that $S \in L^{2,\alpha}$ and taking into account Remark 2.1 we conclude that $\kappa \in H^{0,\alpha}(S)$. Hence in (2.2.3) we may include the point ζ in the contour integration. Thus the integral in (2.2.3) may be interpreted as a proper integral.

The main result of this section is Theorem 2.2. The proof of this theorem leans strongly on $3-D$ results given by Günter [1, p. 312], who has proven the following theorem: *let $S \in L^{2,\alpha}$ and $\mu \in H^{0,\alpha}(S)$, then $\bar{\phi}_d^{(3)} \in H^{1,\alpha}(S)$.* The reason why we cannot quote this theorem is that we only assume μ to be essentially bounded. We define this space of functions on a rectifiable contour, because Priwalow [6] has shown that, for such a contour, measurability and summability can be introduced in the same way as for a straight line.

Definition 4.1.5. $L_\infty(S)$ denotes the Banach space of essentially bounded functions on a rectifiable contour S and these functions are measurable with respect to S. The associated norm is

$$\|\mu\|_\infty = \operatorname*{ess\,sup}_{z \in S} |\mu(z)|.$$

It is noteworthy to remark that if $S \in L^{2,\alpha}$ then S is rectifiable.

Theorem 2.2. *Let $S \in L^{2,\alpha}$ and $\mu \in L_\infty(S)$, then*

$$\bar{\phi}_d \in H^{1,\alpha}(S).$$

Proof. Let (ξ, η) be a local coordinate system about a certain point $P \in S$. Using Definition 2.3 we split the boundary into two parts Σ and $S-\Sigma$. Let $\zeta \in \Sigma$. For (2.2.3) we obtain

$$\bar{\phi}_d(\zeta) = \frac{1}{\pi} \int_{S-\Sigma} \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z + \frac{1}{\pi} \int_\Sigma \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z.$$

In the first integral $z \in S-\Sigma$ and therefore $|z-\zeta| \neq 0$. If we replace ζ by $(\xi, F(\xi))$ we obtain a function of ξ which has bounded and continuous derivatives up to order 2 (since $S \in L^{2,\alpha}$); hence the first integral certainly belongs to the class $H^{1,\alpha}(S)$. We proceed to establish that the second integral also belongs to $H^{1,\alpha}(S)$. We denote the coordinates of the point ζ by ξ, η and those of the integration point z by x,y. Substituting $\eta = F(\xi)$ and $y = F(x)$, we obtain

$$I_2 \equiv \int_\Sigma \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z$$

$$= \int_{\bar{D}_{\epsilon,P}} \mu(x) \frac{F(x) - F(\xi) + (\xi-x)F'(x)}{\{(x-\xi)^2 + (F(x)-F(\xi))^2\}} \, dx.$$

We define the following functions

$$\psi_1(\xi, x) = \int_0^1 F'(\xi + (x-\xi)\,t)\, dt$$

and

$$\psi_2(\xi, x) = \int_0^1 tF''(\xi + (x-\xi)\,t)\, dt.$$

Integrating by parts, we obtain:

$$F(x) - F(\xi) + (\xi-x)F'(x) = -(\xi-x)^2\, \psi_2(\xi, x), \tag{2.3}$$

and

$$F(x) - F(\xi) = (x-\xi)\, \psi_1(\xi, x). \tag{2.4}$$

Hence, the second integral becomes

$$I_2 = \int_{\bar{D}_{\epsilon,P}} \mu(x) \frac{\psi_2(\xi,x)}{(1+\psi_1^2(\xi,x))} \, dx. \tag{2.5}$$

Assuming $\| \mu \|_\infty < A$ we have to prove that:

$$\left| \frac{dI_2}{d\xi} \right| < CA, \tag{2.6}$$

and

$$\left| \frac{d}{d\xi} I_2(\xi_1) - \frac{d}{d\xi} I_2(\xi_2) \right| < CA \, |\xi_1 - \xi_2|^\alpha, \quad \forall \xi_1, \xi_2 \in \bar{D}_{\epsilon,P}. \tag{2.7}$$

First, we show that

$$|\psi_2(\xi_1,x) - \psi_2(\xi_2,x)| \leq C \, |\xi_1 - \xi_2|^\alpha.$$

Indeed, since $F'' \in C^{0,\alpha}(\bar{D}_{\epsilon,P})$, we have:

$$\begin{aligned} |\psi_2(\xi_1,x) - \psi_2(\xi_2,x)| &= \left| \int_0^1 t\{F''(x+(\xi_1-x)t) - F''(x+(\xi_2-x)t)\} \, dt \right| \\ &\leq \int_0^1 |F''(x+(\xi_1-x)t) - F''(x+(\xi_2-x)t)| \, dt \\ &\leq C \int_0^1 |x+(\xi_1-x)t - x + (\xi_2-x)t|^\alpha \, dt \\ &\leq C' \, |\xi_1-\xi_2|^\alpha. \end{aligned}$$

In order to prove (2.6) and (2.7) we first investigate the function

$$\frac{\psi_2(\xi,x)}{1+\psi_1^2(\xi,x)}. \tag{2.8}$$

We denote this function by R. Since ψ_1 and ψ_2 are bounded, it follows that R is bounded too. We shall now prove the following inequalities:

$$\left| \frac{\partial R}{\partial \xi} \right| < \frac{C}{|\xi - x|^{1-\alpha}} \qquad \text{for } \xi \to x, \tag{2.9}$$

$$\left| \frac{\partial^2 R}{\partial \xi^2} \right| < \frac{C}{|\xi - x|^{2-\alpha}} \qquad \text{for } \xi \to x. \tag{2.10}$$

Differentiating R we obtain:

$$\frac{\partial R}{\partial \xi} = \frac{\partial \psi_2}{\partial \xi} (1 + \psi_1^2)^{-1} - 2(1 + \psi_1^2)^{-2} \psi_1 \frac{\partial \psi_1}{\partial \xi} \psi_2. \tag{2.11}$$

It can be easily proved that $|\partial \psi_1 / \partial \xi| < c_0$. Since $|1 + \psi_1^2| > 1$, $|\psi_1| < c_1$ and $|\psi_2| < c_2$ (with c_0, c_1, c_2 certain constants that depend on F), it follows that

$$\left| \frac{\partial R}{\partial \xi} \right| < \left| \frac{\partial \psi_2}{\partial \xi} \right| + 2c_0 c_1 c_2.$$

Inequality (2.9) will have been proved when we have shown that

$$\left| \frac{\partial \psi_2}{\partial \xi} \right| < \frac{C}{|\xi - x|^{1-\alpha}}. \tag{2.12}$$

From (2.3) we obtain

$$\frac{\partial \psi_2}{\partial \xi} = 2 \frac{\psi_2}{(x-\xi)} + \frac{F'(\xi) - F'(x)}{(x-\xi)^2}. \tag{2.13}$$

Since

$$F'(\xi) - F'(x) = -(x-\xi) \int_0^1 F''(\xi + (x-\xi)t)\,\mathrm{d}t,$$

it follows that

$$\frac{\partial \psi_2}{\partial \xi} = \{2 \int_0^1 tF''(\xi + (x-\xi)t)\,\mathrm{d}t - \int_0^1 F''(\xi + (x-\xi)t)\,\mathrm{d}t\}/(x-\xi)$$

$$= \frac{1}{(x-\xi)} \int_0^1 \{F''(\xi + (x-\xi)\sqrt{t}) - F''(\xi + (x-\xi)t)\}\,\mathrm{d}t.$$

Hence,

$$\left| \frac{\partial \psi_2}{\partial \xi} \right| \leq \frac{C}{|x-\xi|} \int_0^1 |\xi + (x-\xi)\sqrt{t} - \xi - (x-\xi)t|^\alpha\,\mathrm{d}t$$

$$\leq \frac{C}{|x-\xi|^{1-\alpha}} \int_0^1 |\sqrt{t} - t|^\alpha\,\mathrm{d}t \leq \frac{c}{|x-\xi|^{1-\alpha}}.$$

Using (2.9) and (2.12) we obtain by another differentiation of R the estimate:

$$\left| \frac{\partial^2 R}{\partial \xi^2} \right| < \left| \frac{\partial^2 \psi_2}{\partial \xi^2} \right| + \frac{c_3}{|x-\xi|^{1-\alpha}} + c_4.$$

Therefore it suffices to show that

$$\left| \frac{\partial^2 \psi_2}{\partial \xi^2} \right| < \frac{c}{|x-\xi|^{2-\alpha}}.$$

Differentiating (2.13) we obtain:

$$\frac{\partial^2 \psi_2}{\partial \xi^2} = \frac{2\psi_2}{(x-\xi)^2} + \frac{2}{(x-\xi)} \frac{\partial \psi_2}{\partial \xi} + \frac{2}{(x-\xi)^3} \{F'(\xi) - F'(x)\} + \frac{F''(\xi)}{(x-\xi)^2}$$

$$= \frac{3}{(x-\xi)} \frac{\partial \psi_2}{\partial \xi} + \frac{1}{(x-\xi)^3} \{F'(\xi) - F'(x)\} + \frac{F''(\xi)}{(x-\xi)^2}$$

$$= \frac{3}{(x-\xi)} \frac{\partial \psi_2}{\partial \xi} + \frac{1}{(x-\xi)^2} \left[F''(\xi) - \int_0^1 F''(\xi + (x-\xi)t)\,\mathrm{d}t \right].$$

Because of the mean value theorem the expression within square brackets is equal to

$$F''(\xi) - F''(\xi + (x-\xi)t^*), \text{ for some } t^* \in [0,1].$$

Since $F'' \in C^{0,\alpha}(\bar{D}_{\epsilon,P})$ and $\left| \frac{\partial \psi_2}{\partial \xi} \right| < \frac{c}{|x-\xi|^{1-\alpha}}$, it follows that

$$\left| \frac{\partial^2 \psi_2}{\partial \xi^2} \right| < \frac{c}{|x-\xi|^{2-\alpha}}.$$

We now consider the integral

$$\frac{dI_2}{d\xi}(\xi) = \int_{\bar{D}_{\epsilon,P}} \mu(x) \frac{\partial R}{\partial \xi}(\xi, x)\,\mathrm{d}x.$$

Without loss of generality we can take $\bar{D}_{\epsilon,P}$ equal to [0,1]. Since $\|\mu\|_\infty \leq A$, it follows that

$$\left| \frac{dI_2}{d\xi} \right| \leq A \int_0^1 \left| \frac{\partial R}{\partial \xi}(\xi, x) \right| \mathrm{d}x.$$

Using estimate (2.9) we conclude that the singularity in $\partial R/\partial \xi$ is integrable. Hence, $dI_2/d\xi$ is bounded. We proceed to establish (2.7). Let $\delta = |\xi_1 - \xi_2|$, then

$$\left| \frac{dI_2}{d\xi}(\xi_1) - \frac{dI_2}{d\xi}(\xi_2) \right| \leq A \int_0^1 \left| \frac{\partial R}{\partial \xi}(\xi_1, x) - \frac{\partial R}{\partial \xi}(\xi_2, x) \right| \mathrm{d}x \tag{2.14}$$

$$\leq A \int_{\xi_1 - 2\delta}^{\xi_1 + 2\delta} \left| \frac{\partial R}{\partial \xi}(\xi_2, x) \right| \mathrm{d}x + A \int_{\xi_1 - 2\delta}^{\xi_1 + 2\delta} \left| \frac{\partial R}{\partial \xi}(\xi_1, x) \right| \mathrm{d}x +$$

$$A \int_0^{\xi_1 - 2\delta} \left| \frac{\partial R}{\partial \xi}(\xi_1, x) - \frac{\partial R}{\partial \xi}(\xi_2, x) \right| \mathrm{d}x + A \int_{\xi_1 + 2\delta}^1 \left| \frac{\partial R}{\partial \xi}(\xi_1, x) - \frac{\partial R}{\partial \xi}(\xi_2, x) \right| \mathrm{d}x$$

Because of inequality (2.9) we obtain for the second integral on the right-hand side of (2.14):

$$\int_{\xi_1-2\delta}^{\xi_1+2\delta} \left|\frac{\partial R}{\partial \xi}(\xi_1)\right| dx < \int_{\xi_1-2\delta}^{\xi_1+2\delta} |\xi_1-x|^{\alpha-1} dx < c(2\delta)^\alpha.$$

Since the interval $|\xi_1-x| < 2\delta$ is contained in the circle $|\xi_2-x| < 3\delta$, we obtain for the first integral the estimate $c\,(3\delta)^\alpha$. Therefore the sum of the first two intgrals is less than a number of the form $c\delta^\alpha$.

Finally, we have to estimate the last two integrals of (2.14). For $x \notin [\xi_1, \xi_2]$ the mean-value theorem yields

$$\frac{\partial R}{\partial \xi}(\xi_1, x) - \frac{\partial R}{\partial \xi}(\xi_2, x) = (\xi_1-\xi_2)\frac{\partial^2 R}{\partial \xi^2}(\xi^*, x),$$

where ξ^* denotes some point of the interval $[\xi_1, \xi_2]$. From inequality (2.10)

$$\left|\frac{\partial R}{\partial \xi}(\xi_1, x) - \frac{\partial R}{\partial \xi}(\xi_2, x)\right| < \delta c\,|\xi^*-x|^{\alpha-2},$$

so that for the third integral the following estimate is obtained:

$$c\delta \int_0^{\xi_1-2\delta} (\xi^*-x)^{\alpha-2} dx = \frac{c\delta}{\alpha-1}\left[(\xi^*-\xi_1+2\delta)^{\alpha-1} - \xi^{*\alpha-1}\right] < \bar{c}\,\delta^\alpha.$$

In the same way we obtain a similar estimate for the last integral. Hence, the left-hand side of (2.14) is less than a number of the form $c\,\delta^\alpha A$. By definition $\delta = |\xi_1-\xi_2|$ and thus inequality (2.7) has been proved. This completes the proof of Theorem 2.2. □

One of the most important applications of potential theory is solving Dirichlet and Neumann problems. The solution of a Dirichlet problem can be sought in the form of the double-layer potential ϕ_d (2.2). In the case of an interior Dirichlet problem the boundary value ϕ_d^- is prescribed and the solution ϕ_d follows from (2.2) as soon as the doublet distribution μ has been determined from equation (2.2.2). Let the integral operator in (2.2.3) be denoted symbolically by K:

$$K\mu(\zeta) = \frac{-1}{\pi}\oint_S \mu(z)\frac{\cos(n_z, z-\zeta)}{|z-\zeta|}\,dS_z, \quad \zeta \in S. \tag{2.15}$$

From Theorem 2.2 it follows that $K\mu \in H^{1,\alpha}(S)$ if $S \in L^{2,\alpha}$ and $\mu \in L_\infty(S)$. We conclude that the operator K maps from the Banach space $L_\infty(S)$ into the class $H^{1,\alpha}(S)$, which is a Banach space too if it is equipped with the following norm:

$$\|f\|_{1,\alpha} = \sum_{l=0}^{1} \|D^l f\|_\alpha,$$

where D denotes differentiation in the tangential direction and

$$\|f\|_\alpha = \|f\|_\infty + \sup_{z_1, z_2 \in S} \frac{|f(z_1) - f(z_2)|}{|z_1-z_2|^\alpha}.$$

Since K is a linear operator, Corollary 2.3 follows directly from Theorem 2.2:
Corollary 2.3. *Let $S \in L^{2,\alpha}$, then the operator K mapping from $L_\infty(S)$ into $H^{1,\alpha}(S)$ is bounded*, i.e.:

$$\| K\mu \|_{1,\alpha} \leq C \, \| \mu \|_\infty, \text{ for all } \mu \in L_\infty(S).$$

We note that the space $H^{1,\alpha}(S)$ is compactly imbedded in the space $L_\infty(S)$. From this property and the previous corollary follows:
Corollary 2.4. *Let $S \in L^{2,\alpha}$, then the operator K mapping from $L_\infty(S)$ into $L_\infty(S)$ is compact.*
Remark 2.2. From the Fredholm alternative theorem for compact operators, it follows that equation (2.2.2) has a unique solution for each boundary function $\phi_d^- \in L_\infty(S)$ if $S \in L^{2,\alpha}$ (see also Zabreyko [9, p. 218]). In addition, the operator $(I-K)^{-1}$ is bounded on the space $L_\infty(S)$,

$$\| (I-K)^{-1} \|_\infty < C.$$

Corollary 2.5. *Let $S \in L^{2,\alpha}$ and $\phi_d^- \in H^{1,\alpha}(S)$, then the solution of (2.2.2) belongs to $H^{1,\alpha}(S)$.*
Proof. From Remark 2.2 we have $\mu \in L_\infty(S)$. But (2.2.2) can be written as

$$\mu = 2\phi_d^- + K\mu.$$

By Theorem 2.2 it follows that the right-hand side belongs to $H^{1,\alpha}(S)$.

3. Aerodynamic application

For incompressible and irrotational flow around a two-dimensional body, there exists a velocity potential ϕ satisfying Laplace's equation

$$\Delta\phi = 0, \tag{3.1}$$

with boundary conditions

$$\frac{\partial\phi}{\partial n_e} = 0 \qquad \text{along the boundary } S, \tag{3.2}$$

and

$$\phi(\zeta) \to U \cdot \zeta \qquad \text{for } |\zeta| \to \infty, \tag{3.3}$$

where $\partial/\partial n_e$ denotes differentiation in the direction of the outward normal to S and $U \cdot \zeta$ denotes the usual inner produce in $\mathbb{R}^2$. The velocity potential ϕ is given by the superposition

$$\phi(\zeta) = \phi_d(\zeta) + U \cdot \zeta, \tag{3.4}$$

where ϕ_d is defined by (2.2) and μ the solution of equation (3.6.1). By standard arguments the above Neumann problem for the exterior of the boundary S is transformed to a Dirichlet problem in the interior.

Lemma 3.1. *Let $S \in L^{2,\alpha}$ and $\mu \in H^{1,\alpha}(S)$, then the boundary condition* (3.2) *can be replaced by:*

$$\phi^-(\zeta) = 0, \quad \zeta \in S. \tag{3.5}$$

Proof. See Martensen [3, p. 247].

From Lemma 3.1 and the equations (3.4) and (3.5), it follows that condition (3.2) is satisfied if the doublet distribution μ is the solution of the following Fredholm integral equation:

$$\mu(\zeta) + \frac{1}{\pi} \oint_S \mu(z) \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z = -2U \cdot \zeta. \tag{3.6.1}$$

We write this equation in operator notation as

$$(I - K)\mu = g, \tag{3.6.2}$$

where $g(\zeta) = -2U \cdot \zeta$ and K the integral operator defined by (2.15). From Remark 2.2 it follows that $(I - K)$ has a bounded inverse on $L_\infty(S)$. Further it can be verified that $g \in H^{1,\alpha}(S)$ and as a consequence of Corollary 2.5 we obtain $\mu \in H^{1,\alpha}(S)$.

Now we discuss the convergence of a sequence of approximations to the unique solution of (3.6.2). We divide the boundary S into N segments S_i, so that $S = S_1 + S_2 + \ldots + S_N$. The begin- and end-points of the i^{th} segment are z_{i-1} and z_i, which are called nodal points. We approximate the function $\mu(\zeta)$, $\zeta \in S$, by a step-function and we solve the resulting equation by a collocation method. The collocation points ζ_i, $i = 1, 2, \ldots, N$, are taken to be the mid-points of the segments S_i.

Definition 3.1. Let (x, y) be a local coordinate system about collocation point ζ_i and let the coordinates of the nodal point z_i be given by $(x_i, F(x_i))$ with $F(x)$ as in Definition 2.3. The coordinates of the collocation point ζ_i are defined by $(\xi, F(\xi))$ with $\xi = (x_{i-1} + x_i)/2$.

We write the approximating step-function in operator notation as follows:

$$T_N\mu(\zeta) = \sum_{i=1}^{N} \mu(\zeta_i) u_i(\zeta) \tag{3.7}$$

with

$$u_i(\zeta) = \begin{cases} 1, & \zeta \in S_i, \\ 0, & \zeta \notin S_i. \end{cases}$$

We define T_N as a linear mapping from the space $C(S)$ of continuous functions on S (with the supremum norm $\|\cdot\|_\infty$) to $X_N = \text{span}(u_1, \ldots, u_N)$. It is noteworthy to remark that T_N is not a projection operator in $C(S)$, because X_N is not a subspace of $C(S)$. However, $X_N \subset L_\infty(S)$ and it is easy to prove the following.

Lemma 3.2. *The mappings T_N and $I - T_N$ are bounded from $C(S)$ into $L_\infty(S)$.*
Proof. For all $f \in C(S)$ we have

$$\sup_{f \in C(S)} \| T_N f \|_\infty = \max_{1 \leq i \leq N} |f(\zeta_i)| \leq \|f\|_\infty . \square$$

Let h_N be a measure of the mesh-size defined by:

$$h_N = \max_{1 \leq i \leq N} |z_i - z_{i-1}|.$$

We assume that the partition of the boundary is such that $\lim_{N \to \infty} h_N = 0$.
Lemma 3.3. *Let $S \in L^{2,\alpha}$ and $f \in H^{1,\alpha}(S)$, then*

$$\| (I - T_N) f \|_\infty \leq C h_N \|f\|_{1,\alpha} \quad \text{as } N \to \infty.$$

Proof. Draw a circle with centre ζ_i and radius h_N. The proof follows from Definition 2.4. □

For a given N an approximate solution of equation (3.6.2) is obtained by solving:

$$(I - T_N K)\mu_N = T_N g, \quad \mu_N \in X_N. \tag{3.8}$$

Remark 3.1. In aerodynamics this collocation method has become very popular because $T_N K \mu_N$ can be easily calculated. In the two-dimensional case, angles have to be measured.
Remark 3.2. As a consequence of Theorem 2.2 an approximation in the space $H^{1,\alpha}(S)$ can be obtained by a single iteration

$$\tilde{\mu}_N = g + K\mu_N, \tag{3.9}$$

where μ_N is the solution of (3.8). It is easily verified that

$$\tilde{\mu}_N = g + K T_N \tilde{\mu}_N.$$

For the case in which T_N represents a projection operator and K a sufficiently regular integral operator, the convergence properties of $\tilde{\mu}_N$ are discussed by Sloan [8].
Remark 3.3. Real aerofoils are given by a data-set of points $\{x_i, y_i\}_{i=1}^M$. Usually a continuous boundary is obtained by a polygon connecting the points of the data-set. However, this polygon does not belong to the class $L^{2,\alpha}$ and a single iteration does not yield an approximation in the space $H^{1,\alpha}(S)$. Therefore, if one wants the approximate solution $\tilde{\mu}_N$ in $H^{1,\alpha}(S)$, it is necessary to construct a smoother boundary through the points $\{x_i, y_i\}$, e.g. a cubic spline approximation so that $S \in L^{2,\alpha}$, except for a small region near the trailing edge.
Lemma 3.4. *Let the finite-dimensional subspace $X_N \subset L_\infty(S)$ be sufficiently large (i.e. the mesh-size of the discretization is sufficiently small) and let $S \in L^{2,\alpha}$. From the existence of a bounded inverse of $I - K$ on $L_\infty(S)$ follow:*

$$(I - T_N K)^{-1} \textit{ exists on } L_\infty(S)$$

and

$$C_1 \equiv \sup_{n \geq N} \| (I - T_n K)^{-1} \|_{L_\infty(S) \to L_\infty(S)} < \infty.$$

Proof. For each $f \in L_\infty(S)$ we have by Lemma 3.3 and Corollary 2.3:

$$\| (I - T_N) Kf \|_\infty \leq c_2 h_N \| Kf \|_{1,\alpha} \leq c_3 h_N \| f \|_\infty .$$

But then $\| K - T_N K \|_{L_\infty(S) \to L_\infty(S)} \to 0$ as $N \to \infty$ and existence and boundedness on $L_\infty(S)$ follow from Neumann's theorem. See also Prenter [5, p. 574]. □

Lemma 3.5. *Let $S \in L^{2,\alpha}$ and $f \in H^{1,\alpha}(S)$, then*

$$\left| \int_{S_i} \{f(z) - f(\zeta_i)\} \, \mathrm{d}S_z \right| \leq C_2 h_N^{2+\alpha} \| f \|_{1,\alpha}, \text{ as } N \to \infty.$$

Proof. Let (x, y) be a local coordinate system about a certain point $P \in S_i$. We denote the coordinates of the point ζ_i by (ξ, η) and those of the integration point z by (x, y). Using Definition 2.4 we can represent $f(z) - f(\zeta_i)$ by

$$\hat{f}(x) - \hat{f}(\xi) = (x - \xi) \hat{f}'(\xi) + (x - \xi) \int_0^1 \{\hat{f}'(\xi + (x - \xi)t) - \hat{f}'(\xi)\} \, \mathrm{d}t.$$

We recall that ζ_i is the midpoint of S_i. Following Definition 3.1 we denote the coordinates of the nodal-point z_i by (x_i, y_i). Let $h = (x_i - x_{i-1})/2$ and $G(x) = \{1 + (F'(x))^2\}^{1/2}$, then the above integral can be estimated as follows:

$$\left| \int_{S_i} \{f(z) - f(\zeta_i)\} \, \mathrm{d}S_z \right| = \left| \int_{\xi - h}^{\xi + h} \{\hat{f}(x) - \hat{f}(\xi)\} G(x) \, \mathrm{d}x \right|$$

$$\leq \left| \int_{\xi - h}^{\xi + h} \{\hat{f}(x) - \hat{f}(\xi)\} \{G(x) - G(\xi)\} \, \mathrm{d}x \right| + \left| \int_{\xi - h}^{\xi + h} (x - \xi) \hat{f}'(\xi) G(\xi) \, \mathrm{d}x \right|$$

$$+ \left| \int_{\xi - h}^{\xi + h} (x - \xi) G(\xi) \int_0^1 \{\hat{f}'(\xi + (x - \xi)t) - \hat{f}'(\xi)\} \, \mathrm{d}t \, \mathrm{d}x \right| .$$

The first part is less than $Ch_N^3 \| f \|_{1,\alpha}$ and the second part is equal to zero. We proceed to estimate the third part. Let $x - \xi = v$. Since G is bounded it follows that

$$I_3 \leq C \left| \int_{-h}^{h} v \int_0^1 \{\hat{f}'(\xi + vt) - \hat{f}'(\xi)\} \, \mathrm{d}t \, \mathrm{d}v \right|$$

$$\leq C \int_{-h}^{h} |v| \int_0^1 |\hat{f}'(\xi + vt) - \hat{f}'(\xi)| \, \mathrm{d}t \, \mathrm{d}v$$

$$\leq C' \int_0^h |v|^{1+\alpha} \| f \|_{1,\alpha} \, \mathrm{d}v \leq C_2 h_N^{2+\alpha} \| f \|_{1,\alpha}. \ \square$$

In Theorem 3.6 we discuss the convergence of the approximate solutions $\mu_N \in X_N$ and $\widetilde{\mu}_N \in H^{1,\alpha}(S)$ to the exact solution μ. We give error estimates for $\|\mu-\mu_N\|_\infty$, $\|T_N\mu-\mu_N\|_\infty$ and $\|\mu-\widetilde{\mu}_N\|_\infty$.

Theorem 3.6. [*Approximation theorem*]. *Let the boundary $S \in L^{2,\alpha}$, then for $N \to \infty$:*

(i) $\|\mu-\mu_N\|_\infty \leq C_3 h_N \|\mu\|_{1,\alpha}$, *where μ is the solution of* (3.6.2) *and μ_N of* (3.8).

(ii) $\|K(I-T_N)f\|_\infty \leq C_4 h_N^{1+\alpha} \|f\|_{1,\alpha}$, *for all $f \in H^{1,\alpha}(S)$.*

(iii) $\|T_N\mu - \mu_N\|_\infty \leq C_5 h_N^{1+\alpha} \|\mu\|_{1,\alpha}$,

(iv) $\|\mu - \widetilde{\mu}_N\|_\infty \leq C_6 h_N^{1+\alpha} \|\mu\|_{1,\alpha}$.

Proof.

(i) From (3.6.2) and (3.8) we get:

$$(I - T_N K)(\mu - \mu_N) = \mu - T_N K\mu - T_N g = \mu - T_N\mu.$$

Use Lemma 3.4 and 3.3 to obtain:

$$\|\mu-\mu_N\|_\infty \leq C_1 \|\mu-T_N\mu\|_\infty \leq C_3 h_N \|\mu\|_{1,\alpha}.$$

(ii) From the construction of $T_N f$ it follows that

$$K(I - T_N)f(\zeta) = -\sum_{i=1}^{N} \int_{S_i} \{f(z) - f(\zeta_i)\} \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z.$$

Without loss of generality we can take $\zeta \in S_i$. Taking into account Remark 2.1 we estimate the (i)-th part of the above sum by:

$$\left| \int_{S_i} \{f(z) - f(\zeta_i)\} \frac{\cos(n_z, z-\zeta)}{|z-\zeta|} \, dS_z \right|$$

$$\leq C \int_{S_i} |f(z) - f(\zeta_i)| \, dS_z \leq C' \int_{\xi-h}^{\xi+h} |x| \, \|f\|_{1,\alpha} \, dx \leq C'' h_N^2 \|f\|_{1,\alpha},$$

where ξ, h are defined in the proof of Lemma 3.5. In the other parts of the above sum we replace z by $(x, F(x))$. Since $S \in L^{2,\alpha}$ the kernel-function $\cos(n_z, z-\zeta)/|z-\zeta|$ has bounded and continuous derivatives up to order 2 with respect to x. Hence, this function can be written as a series expansion involving powers of $(x-\xi)$. Applying Lemma 3.5 we obtain

$$\|K(I - T_N)f\|_\infty \leq C'' h_N^2 \|f\|_{1,\alpha} + (N-1) C_2 h_N^{2+\alpha} \|f\|_{1,\alpha}$$

$$\leq C_4 h_N^{1+\alpha} \|f\|_{1,\alpha}.$$

(iii) From equation (3.6.2) we get

$$(I - T_N K) T_N\mu = T_N g + T_N K(\mu - T_N\mu)$$

and subtract (3.8) to obtain

$$(I - T_N K)(T_N \mu - \mu_N) = T_N K(\mu - T_N \mu).$$

Applying Lemma 3.4 and 3.2 we have

$$\| T_N \mu - \mu_N \|_\infty \leq C \| K(\mu - T_N \mu) \|_\infty .$$

Since $\mu \in H^{1,\alpha}(S)$ the proof follows from part (ii) of this theorem.

(iv) From (3.6.2) and (3.9) it follows that

$$\| \mu - \tilde{\mu}_N \|_\infty = \| K(\mu - \mu_N) \|_\infty$$

$$\leq \| K(\mu - T_N \mu) \|_\infty + \| K(T_N \mu - \mu_N) \|_\infty .$$

Using part (ii) and (iii) we obtain the proof of (iv). □

With respect to the smoothness of the boundary S, part *(ii)* of Theorem 3.6 is a modification of results given by Kantorowitsch [2, p. 127]. He has proven the following: let the boundary S be given by the parametric equations

$$z(t) = X(t) + iY(t), \quad t \in [0,1],$$

and let $\omega(s, t) = \arg(z(s) - z(t))$. If ω is three times continuously differentiable with respect to s (this assumption is stronger than $S \in L^{2,\alpha}$!) and the function f is two times continuously differentiable (i.e. $f \in C^{(2)}[0,1]$), then

$$\| K(I - T_N) f \|_\infty \leq C h_N^2 \| f \|_2 ,$$

where $\| \cdot \|_2$ is the usual norm of the space $C^{(2)}[0,1]$.

Usually part *(iii)* of Theorem 3.6 is called super-convergence on the collocation points. Performing a single iteration of type (3.9) the order of super-convergence is extended to all points of the boundary, as has been shown by part *(iv)*.

So far we did not say anything about how to solve equation (3.8). When the dimension of X_N is small it can be solved by a direct method (e.g. Gaussian elimination). However, when the dimension is large one usually uses iterative techniques. In [7] we have applied a multiple-grid iterative process to (3.8) and we have estimated its reduction factor η by

$$\eta = \| (I - T_N) K \|_{X_N \to X_N} .$$

Using Corollary 2.3 and Lemma 3.3 we obtain that $\eta < C h_N$, as $N \to \infty$. Indeed, the reduction factor decreases as N increases. Then, asymptotically for $N \to \infty$, the multiple-grid method needs only 2 iterations to obtain a result for which the superconvergence on the collocation points is preserved.

We have applied the numerical method of this section to the calculation of non-circulatory, potential flow around a Kármán-Trefftz aerofoil. This aerofoil is obtained from the circle in the x-plane, $x = c\, e^{i\theta}$, by means of the mapping

$$z = f(x) = (x - x_t)^k / (x - c(\delta - i\gamma))^{k-1}, \tag{3.10}$$

where k measures the trailing edge angle, γ the camber and δ the thickness of the aerofoil;

$$c = 2l(\delta + \sqrt{1-\gamma^2})^{k-1} / (2\sqrt{1-\gamma^2})^k, \quad x_t = c(\sqrt{1-\gamma^2} - i\gamma),$$

with l the length of the aerofoil. To make f single-valued we take the principal value in (3.10).

The Kármán-Trefftz aerofoil does not belong to the class $L^{2,\alpha}$ because of the presence of the trailing edge at $z = z_t$. At this point the curvature is not defined. In the present paper we remove the corner by the additional mapping

$$\omega = g(z) = z(1 - \tilde{z}/z)^{1-1/k}, \tag{3.11}$$

where $\tilde{z}$ is a point inside the aerofoil. Here we locate it arbitrarily at $\tilde{z} = -1.95$. By means of (3.11) the aerofoil in the z-plane is converted into a quasi-circular shape in the ω-plane. This has been done because the inverse mapping of (3.10) converts real aerofoils (which do not belong to the family of Kármán-Trefftz aerofoils) into quasi-circular shapes too.

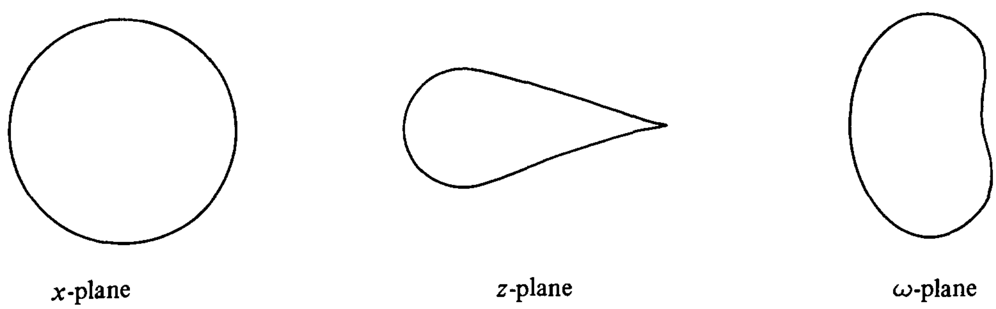

Figure 3.1.

We divide the circle in the x-plane uniformly into N segments. Hence the nodal-points in this plane are given by:

$$x_j = ce^{i\theta_j}, \theta_j = 2\pi j/N, \quad j = 0, 1, ..., N.$$

Substituting x_j, $j = 0, 1, ..., N$, into (3.10) and (3.11) successively, we obtain the nodal-points $\{z_j\}$ in the z-plane and $\{\omega_j\}$ in the ω-plane.

The tangential velocity V_j at the point z_j ($z_j \in K.T.$-aerofoil) is obtained numerically by:

$$V_j = \frac{|\mu_{N,j+1} - \mu_{N,j}|}{|\omega_{j+\frac{1}{2}} - \omega_{j-\frac{1}{2}}|} * \left| \frac{d\omega}{dz} \right|_{z=z_j}, \quad j = 1, ..., N-1,$$

where $\omega_{j+\frac{1}{2}}$ is the collocation point corresponding with $x_{j+\frac{1}{2}}$. We note that V_j is obtained by numerical differentiation. Therefore, from Theorem 3.6 *(iv)* we obtain the following error estimate

$$\max_{1\leq j\leq N-1} |V_j - V_{\text{exact}}(z_j)| \leq ch_N^{\alpha}, \quad \text{as} \quad N\to\infty. \tag{3.12}$$

We have taken the following test cases:

(a) $k = 1.90$, $\delta = 0.05$, $l = 1.0$, $\gamma = 0.0$, $U = (1.0, 0.0)$,

(b) $k = 1.99$, $\delta = 0.05$, $l = 1.0$, $\gamma = 0.0$, $U = (1.0, 0.0)$.

In Table 1 we give the maximum error in the tangential velocity, i.e. the left-hand side of (3.12), for increasing values of N. For the above test cases the error estimate (3.12) is found to be too pessimistic.

N \ k	1.90	1.99
32	.12 (−1)	.53 (−1)
64	.62 (−2)	.26 (−1)
128	.16 (−2)	.66 (−2)
256	.39 (−3)	.19 (−2)

Table 1. Maximum error in tangential velocity ($\delta = 0.05$, $l = 1.0$, $\gamma = 0.0$, $U = (1.0, 0.0)$).

For $N = 64$, 128 and 256 the above results have been obtained by a multiple-grid iterative process.

Acknowledgements

The author wishes to express his gratitude to prof. dr. P. Wesseling for his continuous interest in this work and dr. O. Diekmann for his helpful suggestions.

REFERENCES

[1] N. M. Günter, *Potential theory and its applications to basic problems of mathematical physics*, F. Ungar Publ., New York, 1967.

[2] L. W. Kantorowitsch and W. I. Krylow, *Näherungsmethoden der höheren Analysis*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.

[3] E. Martensen, Berechnung der Druckverteilung an Gitterprofilen in ebener Potentialströming mit einer Fredholmschen Integralgleichung, *Arch. Rat. Mech. Anal.* 3 (1959) 235–270.

[4] N. I. Muschelischwili, *Singuläre Integralgleichungen*, Akademie-Verlag, Berlin, 1965.

[5] P. M. Prenter, A collocation method for the numerical solution of integral equations, *SIAM J. Numer. Anal.* 10 (1973) 570–581.
[6] L. L. Priwalow, *Randeigenschaften analytischer Funktionen*, VEB Deutscher Verlag der Wissenschaften, Berlin, 1956.
[7] H. Schippers, *The automatic solution of Fredholm equations of the second kind*, Rep. NW 99/80, Mathematisch Centrum, Amsterdam, 1980 (preprint).
[8] I. H. Sloan, E. Noussair and B. J. Burn, Projection methods for equations of the second kind, *J. Meth. Anal. and Appl.* 69 (1979) 84–103.
[9] P. P. Zabreyko et al., *Integral equations – a reference text*, Noordhoff International Publ., Leiden, 1975.